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SELF-SIMILAR SOLUTIONS OF THE DYNAMICS EQUATIONS OF AN IDEAL ELASTIC-PLASTIC BODY UNDER TRESCA PLASTICITY CONDITIONS

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Self-similar solutions of the dynamics equations of an ideal elastic-plastic body under Mises plasticity conditions were examined in [1-4], where the solutions of the boundary-value problems were reduced to the solution of two-point problems of nonlinear differential equations with singularities. It is shown below that these equations, under Tresca plasticity conditions, are integrated in quadratures which will permit achievement of significant simplifications.

1. The equations describing the behavior of Prandtl-Reiss bodies have the form

$$e_{ij}=e^p_{ij}+e^e_{ij},\ e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}),\ \ \sigma_{ij}=\lambda e_{kk}\delta_{ij}+2\mu e^e_{ij}; \tag{1.1}$$

$$\sigma_{ij,j}-\rho\dot{v}_i=0,\ v_i=\dot{u}_i, \tag{1.2}$$

where u_i is the displacement, and the dot denotes differentiation with respect to time.

Let us consider the plane strain of a medium under Tresk plasticity conditions

$$|\tau_{\max}|=k. \tag{1.3}$$

Let σ_{33} be the third principal stress σ_3. We designate the other two such that $\sigma_1>\sigma_2$. Let us examine possible variations.

A. Let $\sigma_1>\sigma_3>\sigma_2$. Then condition (1.3) takes the form

$$\sigma_1-\sigma_2=2k. \tag{1.4}$$

From the associated flow law there follows

$$\dot{e}^p_1+\dot{e}^p_2=0,\ \ \dot{e}^p_3=0,\ \ \dot{e}^p_1>0.$$

Since $e_{33}=0$, $e^e_{33}=0$, we have from (1.1)

$$\sigma_3=\sigma_{33}=(1/2)(\sigma_{11}+\sigma_{22})\lambda(\lambda+\mu)^{-1}.$$

We have for the principal stresses in the Ox_1x_2 plane

$$\sigma_1,\sigma_2=\sigma\pm\sqrt{(\sigma_{11}-\sigma_{22})^2+4\sigma^2_{12}}. \tag{1.5}$$

From the relationships (1.4) and (1.5) there follows

$$(\sigma_{11}-\sigma_{22})^2+4\sigma^2_{12}=4k^2. \tag{1.6}$$

The associated flow law has the form

$$\frac{\sigma_{11}-\sigma_{22}}{2\sigma_{12}}=\frac{\dot{e}^p_{11}-\dot{e}^p_{22}}{2\dot{e}^p_{12}},\ \ \dot{e}^p_{11}+\dot{e}^p_{22}=0. \tag{1.7}$$

We satisfy condition (1.6) by setting

$$\sigma_{11}=\sigma+k\cos 2\theta,\ \sigma_{22}=\sigma-k\cos 2\theta,\ \sigma_{12}=k\sin 2\theta, \tag{1.8}$$

where θ is the angle between the first principal direction and the Ox_1 axis. Substituting (1.8) into the relations (1.1), (1.2), and (1.7), we obtain a system of equations to determine σ, θ, v_1, v_2

$$\sigma_{,1}-2k(\theta_{,1}\sin 2\theta-\theta_{,2}\cos 2\theta)-\rho\dot{v}_1=0; \tag{1.9}$$

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$$\sigma_{,2} - 2k(\theta_{,2}\sin 2\theta + \theta_{,1}\cos 2\theta) - \rho\dot{v}_2 = 0; \quad (1.10)$$

$$2k\dot{\theta} = \mu(v_{2,2} - v_{1,1})\sin 2\theta + \mu(v_{1,2} + v_{2,1})\cos 2\theta; \quad (1.11)$$

$$\dot{\sigma} = (\lambda + \mu)(v_{1,1} + v_{2,2}). \quad (1.12)$$

The principal values of the strain rates are determined from the formulas

$$\dot{e}_1^p = \dot{e}_{11}^p \cos^2\theta + \dot{e}_{22}^p \sin^2\theta + 2\dot{e}_{12}^p \sin\theta\cos\theta,$$
$$\dot{e}_2^p = \dot{e}_{11}^p \sin^2\theta + \dot{e}_{22}^p \cos^2\theta - 2\dot{e}_{12}^p \sin\theta\cos\theta.$$

The relationships (1.9)-(1.12) hold under the conditions $\sigma_1 > \sigma_3 > \sigma_2$ and $\dot{e}_1^p > 0$. These conditions have the form

$$(\lambda + \mu)k > \mu\sigma > -(\lambda + \mu)k; \quad (1.13)$$

$$(v_{2,2} - v_{1,1})\cos 2\theta - (v_{1,2} + v_{2,1})\sin 2\theta < 0. \quad (1.14)$$

B. Let $\sigma_3 = \sigma_2$. Then the stress state corresponds to intersection of two faces of the plasticity condition

$$\sigma_1 - \sigma_2 = 2k,\ \sigma_1 - \sigma_3 = 2k. \quad (1.15)$$

There follows from the generalized associated flow law

$$\dot{e}_1^p + \dot{e}_2^p + \dot{e}_3^p = 0,\ \ \dot{e}_1^p > 0,\ \ \dot{e}_2^p < 0,\ \ \dot{e}_3^p < 0.$$

We obtain from (1.1) and (1.15)

$$\dot{\sigma}_1 + \dot{\sigma}_2 + \dot{\sigma}_3 = (3\lambda + 2\mu)(v_{1,1} + v_{2,2}). \quad (1.16)$$

From (1.15) we have $\dot{\sigma}_1 = \dot{\sigma}_2 = \dot{\sigma}_3 = \sigma$ and (1.16) takes the form

$$\dot{\sigma} = (\lambda + (2/3)\mu)(v_{1,1} + v_{2,2}). \quad (1.17)$$

Let us note that the first condition in (1.15) and condition (1.4) agree and by reasoning and performing calculations as in case A, we arrive at the deduction that σ, θ, v_1, v_2 satisfy equations (1.9)-(1.11) on the edge of the Tresca plasticity conditions, while (1.12) is replaced by (1.17).

The inequalities $\dot{e}_2^p < 0$, $\dot{e}_3^p < 0$, i.e.,

$$(v_{1,1} - v_{2,2})\cos 2\theta + (v_{1,2} + v_{2,1})\sin 2\theta \geqslant (1/3)(v_{1,1} + v_{2,2}) \geqslant 0$$

should hold on the edge under consideration.

C. Let $\sigma_1 = \sigma_3$. Then the stress state corresponds to an edge of the Tresk condition, which is the intersection of the faces

$$\sigma_1 - \sigma_2 = 2k,\ \sigma_3 - \sigma_2 = 2k. \quad (1.18)$$

There follows from the generalized associated flow law

$$\dot{e}_1^p + \dot{e}_2^p + \dot{e}_3^p = 0,\ \ \dot{e}_1^p > 0,\ \ \dot{e}_2^p < 0,\ \ \dot{e}_3^p > 0.$$

Reasoning as in case B, we obtain that σ, θ, v_1, v_2 satisfy the equations (1.9)-(1.11), (1.17) on the edge σ under consideration, while the inequalities $\dot{e}_1^p > 0$, $\dot{e}_3^p > 0$ take the form

$$(v_{1,1} - v_{2,2})\cos 2\theta + (v_{1,2} + v_{2,1})\sin 2\theta \geqslant -(1/3)(v_{1,1} + v_{2,2}) \geqslant 0. \quad (1.19)$$

D. Let $\sigma_1 > \sigma_2 > \sigma_3$; then the plasticity condition has the form

$$\sigma_1 - \sigma_3 = 2k. \quad (1.20)$$

From the associated flow law we obtain

$$\dot{e}_1^p + \dot{e}_3^p = 0,\ \ \dot{e}_2^p = 0,\ \ \dot{e}_1^p > 0,\ \ \dot{e}_3^p < 0. \quad (1.21)$$

The stresses and plastic strain rates are expressed in terms of the principal values by means of the formulas

$$\sigma_{11} = \sigma + \gamma\sin^2\theta,\ \sigma_{22} = \sigma + \gamma\cos^2\theta,\ \sigma_{12} = -\gamma\cos\theta\sin\theta,$$
$$\gamma = \sigma_2 - \sigma_1,\ \ \dot{e}_{11}^p = \dot{e}_1^p\cos^2\theta,\ \ \dot{e}_{22}^p = \dot{e}_1^p\sin^2\theta,\ \ \dot{e}_{12}^p = \dot{e}_1^p\sin\theta\cos\theta \quad (1.22)$$

We obtain from the motion equations

$$\sigma_{1,1} + \gamma_{,1} \sin^2\theta - \gamma_{,2} \sin\theta\cos\theta + \gamma(\theta_{,1}\sin 2\theta - \theta_{,2}\cos 2\theta) = \rho\dot{v}_1,$$
$$\sigma_{1,2} + \gamma_{,2}\cos^2\theta - \gamma_{,1}\sin\theta\cos\theta - \gamma(\theta_{,2}\sin 2\theta + \theta_{,1}\cos 2\theta) = \rho\dot{v}_2. \quad (1.23)$$

Differentiating (1.1) with respect to time and substituting (1.22), we obtain after eliminating $\dot{e}_1^p$ and $\dot{e}_3^p$

$$3\dot{\sigma}_1 + \dot{\gamma} = (3\lambda + 2\mu)(v_{1,1} + v_{2,2}),$$
$$(\dot{\sigma}_1 - \dot{\gamma})\sin 2\theta - 2\gamma\dot{\theta}\cos 2\theta = \lambda(v_{1,1} + v_{2,2})\sin 2\theta + 2\mu(v_{1,2} + v_{2,1}), \quad (1.24)$$
$$\gamma\dot{\theta} + \mu(v_{2,2} - v_{1,1})\sin 2\theta + \mu(v_{1,2} + v_{2,1})\cos 2\theta = 0.$$

Therefore, to determine the five unknown functions σ_1, γ, θ, v_1, v_2 on the face (1.20) we have a system of five equations (1.23) and (1.24). By using (1.1), we obtain from (1.21) that (1.23) and (1.24) hold under the conditions

$$(v_{2,2} - v_{1,1})\cos 2\theta - (v_{1,2} + v_{2,1})\sin 2\theta \leqslant v_{1,1} + v_{2,2}, \quad -2k < \gamma < 0.$$

E. Let $\sigma_3 > \sigma_1 > \sigma_2$; then the plasticity condition has the form

$$\sigma_3 - \sigma_2 = 2k.$$

We obtain from the associated flow law

$$\dot{e}_2^p + \dot{e}_3^p = 0, \quad \dot{e}_1^p = 0, \quad \dot{e}_2^p < 0, \quad \dot{e}_3^p > 0.$$

We have for the stress and plastic strain rate components

$$\sigma_{11} = \sigma_2 - \gamma\cos^2\theta, \quad \sigma_{22} = \sigma_2 - \gamma\sin^2\theta, \quad \sigma_{12} = -\gamma\cos\theta\sin\theta,$$
$$\dot{e}_{11}^p = \dot{e}_2^p\sin^2\theta, \quad \dot{e}_{22}^p = \dot{e}_2^p\cos^2\theta, \quad \dot{e}_{12}^p = -\dot{e}_2^p\sin\theta\cos\theta.$$

Furthermore, by reasoning as in case D, we obtain a system of equations to determine σ_2, θ, γ, v_1, v_2 in the form

$$\sigma_{2,1} - \gamma_{,1}\cos^2\theta - \gamma_{,2}\sin\theta\cos\theta + \gamma(\theta_{,1}\sin 2\theta - \theta_{,2}\cos 2\theta) = \rho\dot{v}_1,$$
$$\sigma_{2,2} - \gamma_{,2}\sin^2\theta - \gamma_{,1}\sin\theta\cos\theta - \gamma(\theta_{,2}\sin 2\theta + \theta_{,1}\cos 2\theta) = \rho\dot{v}_2,$$
$$(\dot{\sigma}_2 + \dot{\gamma})\sin 2\theta + 2\gamma\dot{\theta}\cos 2\theta + 2\mu(v_{2,1} + v_{1,2}) = \lambda(v_{1,1} + v_{2,2})\sin 2\theta, \quad (1.25)$$
$$\gamma\dot{\theta} + \mu(v_{2,2} - v_{1,1})\sin 2\theta + \mu(v_{1,2} + v_{2,1})\cos 2\theta = 0,$$
$$3\dot{\sigma}_2 - \dot{\gamma} = (3\lambda + 2\mu)(v_{1,1} + v_{2,2}),$$

which will hold under the conditions

$$(v_{2,2} - v_{1,1})\cos 2\theta - (v_{1,2} + v_{2,1})\sin 2\theta \geqslant -(v_{1,1} + v_{2,2}), \quad -2k < \gamma < 0.$$

F. Let $\sigma_1 = \sigma_2$. Then $\sigma_{11} = \sigma_{22} = \sigma$, $\sigma_{12} = 0$ and there follows from the motion equations

$$\sigma_{,1} - \rho\dot{v}_1 = 0, \quad \sigma_{,2} - \rho\dot{v}_2 = 0. \quad (1.26)$$

The plasticity condition will be satisfied if

$$\sigma_3 - \sigma_1 = 2k, \quad \sigma_3 - \sigma_2 = 2k. \quad (1.27)$$

There follows from the associated flow law

$$\dot{e}_1^p + \dot{e}_2^p + \dot{e}_3^p = 0, \quad \dot{e}_1^p < 0, \quad \dot{e}_2^p < 0, \quad \dot{e}_3^p > 0. \quad (1.28)$$

There follows $\dot{\sigma}_3 = \dot{\sigma}_{23} = \dot{\sigma}$ from (1.27) and we obtain from (1.1)

$$\sigma = (\lambda + (2/3)\mu)(v_{1,1} + v_{2,2}). \quad (1.29)$$

The relationships (1.26) and (1.29) are a closed linear system of equations to determine σ, v_1, v_2. The inequalities (1.28) will be satisfied for

$$(1/3)(v_{1,1} + v_{2,2}) + \sqrt{(v_{1,1} - v_{2,2})^2 + (v_{1,2} + v_{2,1})^2} < 0.$$

If the plasticity condition has the form $\sigma_1 - \sigma_3 = 2k$, $\sigma_2 - \sigma_3 = 2k$, then (1.29) holds, and the associated flow law

$$\dot{e}_1^p + \dot{e}_2^p + \dot{e}_3^p = 0, \quad \dot{e}_1^p > 0, \quad \dot{e}_2^p > 0, \quad \dot{e}_3^p < 0$$

will be satisfied, if

$$(1/3)(v_{1,1} + v_{2,2}) - \sqrt{(v_{1,1} - v_{2,2})^2 + (v_{1,2} + v_{2,1})^2} > 0.$$

Therefore, in contrast to the rigidly plastic problem, the Tresca plasticity condition edges play the same role in the plane elastic-plastic problem as do the faces, i.e., result in a closed system of equations in the presence of constraints, where this system holds. The idea of using the edges of the Tresca plasticity conditions to solve axisymmetric boundary-value problems was first utilized in [5], this idea was later used extensively in [6] to solve three-dimensional problems. Definite simplifications can be achieved even when solving plane elastic-plastic problems.

2. Let us examine the self-similar solution of the equations for the plane dynamics problem of an elastic-plastic body under Tresca plasticity conditions.

We take the angle

$$\varphi = \operatorname{arc\,tg}\, [x_2(x_1 - ct)^{-1}],$$

where $c = \text{const}$, $c > c_1 = \sqrt{(\lambda + 2\mu)\rho^{-1}}$, as the self-similar variable.

Going over to the self-similar variable in (1.9)-(1.12), we obtain

$$\begin{gathered} -\sigma' \sin\varphi + 2k\theta' \cos(\varphi - 2\theta) - \rho c v_1' \sin\varphi = 0, \\ \sigma' \cos\varphi - 2k\theta' \sin(\varphi - 2\theta) - \rho c v_2' \sin\varphi = 0, \\ c\sigma' \sin\varphi + (\lambda + \mu)\left(v_1' \sin\varphi - v_2' \cos\varphi\right) = 0, \\ 2k\theta' c \sin\varphi + \mu v_1' \cos(\varphi - 2\theta) - \mu v_2' \sin(\varphi - 2\theta) = 0. \end{gathered} \tag{2.1}$$

The system (2.1) has trivial solutions, i.e., σ, θ, v_1, v_2 are constants independent of φ.

Other solutions of this system are possible only under the condition that its determinant vanishes:

$$\begin{gathered} p^2(1 - p^2)\cos^2 2(\varphi - \theta) = M^2 \sin^2\varphi(1 - M^2 \sin^2\varphi), \\ M = cc_1^{-1}, \quad p = c_2 c_1^{-1}, \quad c_2^2 = \mu\rho^{-1}. \end{gathered} \tag{2.2}$$

Since $0 < p^2 < 1/2$, real values of θ can be obtained from (2.2) only for φ, satisfying the inequalities

$$0 \leqslant M^2 \sin^2\varphi \leqslant p^2, \quad 1 - p^2 \leqslant M^2 \sin^2\varphi \leqslant 1. \tag{2.3}$$

Differentiating (2.2) with respect to φ and eliminating θ' from (2.1), after integrating we can represent the solution in the form

$$p\frac{\sigma}{2k} = \operatorname{sign}(\cos 2(\varphi - \theta))\left[\frac{1 - 2p^2}{4p} \times \right. \\ \left. \times \ln\left|\frac{pM\sin\varphi - \sqrt{(1 - p^2)(1 - M^2\sin^2\varphi)}}{pM\sin\varphi + \sqrt{(1 - p^2)(1 - M^2\sin^2\varphi)}}\right| + \sqrt{1 - p^2}\arcsin(M\sin\varphi)\right] - \\ - \operatorname{sign}(\sin 2(\varphi - \theta))\sqrt{1 - p^2}\int \frac{\sqrt{1 - M^2\sin^2\varphi}}{\sqrt{1 - p^2 - M^2\sin^2\varphi}}\,d\varphi + C_1; \tag{2.4}$$

$$\frac{v_2 \rho c p\sqrt{1 - p^2}}{k} = -\operatorname{sign}(\sin 2(\varphi - \theta))\left[2\sqrt{(p^2 - M^2\sin^2\varphi)(1 - p^2 - M^2\sin^2\varphi)} - \right. \\ \left. - (1 - 2p^2)\ln\left|\sqrt{p^2 - M^2\sin^2\varphi} + \sqrt{1 - p^2 - M^2\sin^2\varphi}\right|\right] - \\ - \operatorname{sign}(\cos 2(\varphi - \theta))\left[(2p^2 - M^2)\ln\left|M\cos\varphi + \sqrt{1 - M^2\sin^2\varphi}\right| + \right. \\ \left. + 2p\sqrt{M^2 + p^2 - 1}\ln\left|\frac{M\cos\varphi - \sqrt{(M^2 + p^2 - 1)(1 - M^2\sin^2\varphi)}}{M\cos\varphi + \sqrt{(M^2 + p^2 - 1)(1 - M^2\sin^2\varphi)}}\right| + 2M\cos\varphi\sqrt{1 - M^2\sin^2\varphi}\right] + C_2. \tag{2.5}$$

It follows from (2.2) that the inequality (1.14) will always be satisfied for values of M and φ belonging to the interval (2.3).

An analysis of the self-similar solutions for the edges (1.15) is performed analogously. The condition that the determinant vanish has the form

$$p^2(1 - (4/3)p^2)\cos^2 2(\varphi - \theta) = (1 - (1/3)p^2 - M^2\sin^2\varphi)M^2\sin^2\varphi. \tag{2.6}$$

Real values of θ can be determined for values of φ, satisfying the inequalities

$$p^2 \geqslant \frac{3}{7}, \quad 0 \leqslant M^2 \sin^2 \varphi \leqslant 1 - \frac{4}{3} p^2, \quad p^2 \leqslant M^2 \sin^2 \varphi \leqslant 1 - \frac{1}{3} p^2,$$
$$p^2 < \frac{3}{7}, \quad 0 \leqslant M^2 \sin^2 \varphi \leqslant p^2, \quad 1 - \frac{4}{3} p^2 \leqslant M^2 \sin^2 \varphi \leqslant 1 - \frac{1}{3} p^2. \tag{2.7}$$

We obtain the expressions for σ, v_2 in the form

$$\frac{\sqrt{1 - \frac{4}{3} p^2}}{2kp} \sigma = \operatorname{sign}(\cos 2(\varphi - \theta)) \left[\frac{1 - \frac{7}{3} p^2}{4p\sqrt{1 - \frac{4}{3} p^2}} \times \right.$$
$$\times \ln \left| \frac{pM \sin \varphi - \sqrt{\left(1 - \frac{4}{3} p^2\right)\left(1 - \frac{p^2}{3} - M^2 \sin^2 \varphi\right)}}{pM \sin \varphi + \sqrt{\left(1 - \frac{4}{3} p^2\right)\left(1 - \frac{p^2}{3} - M^2 \sin^2 \varphi\right)}} \right| + \left. \arcsin\left(\frac{M \sin \varphi}{\sqrt{1 - \frac{p^2}{3}}} \right) \right] + \tag{2.8}$$
$$+ \operatorname{sign}(\sin 2(\varphi - \theta)) \int \frac{p^2 - M^2 \sin^2 \varphi}{\sqrt{1 - \frac{4}{3} p^2 - M^2 \sin^2 \varphi}} d\varphi + C_3;$$

$$\frac{\rho c}{2k} v_2 = -\frac{\operatorname{sign}(\sin 2(\varphi - \theta))}{2p\sqrt{1 - \frac{4}{3} p^2}} \left[\ln \left| \frac{1 + t}{1 - t} \right| + p^2 \ln \left| \frac{p^2 t^2 - 1 + \frac{4}{3} p^2}{p^2 t^2 - 1} \right| + \right.$$
$$\left. + \frac{1 - \frac{7}{3} p^2}{2(t^2 - 1)} - \frac{p}{\sqrt{1 - \frac{4}{3} p^2}} \ln \left| \frac{pt + \sqrt{1 - \frac{4}{3} p^2}}{pt - \sqrt{1 - \frac{4}{3} p^2}} \right| \right] - \frac{\operatorname{sign}(\cos 2(\varphi - \theta))}{p\sqrt{1 - \frac{4}{3} p^2}} \times$$
$$\times \ln \left| M \cos \varphi + \sqrt{1 - \frac{1}{3} p^2 - M^2 \sin^2 \varphi} \right| + \operatorname{sign}(\cos 2(\varphi - \theta)) \frac{\sqrt{1 - \frac{4}{3} p^2}}{2p} \times \tag{2.9}$$
$$\times \left(1 - \frac{1}{3} p^2 - 2M^2\right) \left\{ \left(1 - \frac{1}{3} p^2 - 2M^2\right)^{-1} \ln \left| \frac{1 + t}{1 - t} \right| + \right.$$
$$+ \frac{1 + M^2\left(\frac{7}{3} p^2 - 1\right) - \frac{13}{3} p^2 + \frac{28}{9} p^4}{2p\left(1 - \frac{4}{3} p^2\right)\left(1 - \frac{1}{3} p^2 - M^2\right)\sqrt{M^2 + \frac{4}{3} p^2 - 1}} \ln \left| \frac{p + t\sqrt{M^2 + \frac{4}{3} p^2 - 1}}{p - t\sqrt{M^2 + \frac{4}{3} p^2 - 1}} \right| +$$
$$\left. + \frac{M\left(1 - \frac{1}{3} p^2\right)}{2\left(1 - \frac{4}{3} p^2\right)\left(M^2 + \frac{p^2}{3} - 1\right)} \ln \left| \frac{Mt + \sqrt{1 - \frac{4}{3} p^2}}{Mt - \sqrt{1 - \frac{4}{3} p^2}} \right| \right\} + C_4,$$

where

$$t^2 = \left(1 - \frac{4}{3} p^2 - M^2 \sin^2 \varphi\right)(p^2 - M^2 \sin^2 \varphi)^{-1}.$$

3. Let an acute wedge with aperture 2α move at a constant supersonic speed c in an elastic-plastic medium. Assuming no contact friction, we write the boundary conditions in the form

$$\sigma_{12}(x_1, 0) = 0, \; v_2(x_1, 0) = -c \operatorname{tg} \alpha, \; x_1 < 0. \tag{3.1}$$

The medium is not perturbed in the domain $a'Oa$ (Fig. 1) ahead of the wedge, i.e., $\sigma_{ij} = 0$, $v_i = 0$. The lines Oa and Oa' move in the direction of the normal at the velocity of the irrotational waves $c_1^2 = (\lambda + 2\mu)/\rho$, which is the maximum possible velocity of perturbation propagation in an elastic-plastic medium [7].

It is shown in [7] that neutral strong-discontinuity waves exist in elastic-plastic medium, on which the following relationships are satisfied

$$\rho c_1^2 = \lambda + 2\mu, \quad [v_i] = \omega \nu_i^1, \quad [e_{ij}^p] = 0, \quad -c_1[\sigma_{ij}] = (\lambda \delta_{ij} + 2\mu \nu_i^1 \nu_j^1)\omega; \tag{3.2}$$

$$\rho c_2^2 = \mu, \quad [v_k]\nu_k^2 = 0, \quad [e_{ij}^p] = 0, \quad -c_2[\sigma_{ij}] = \mu([v_i]\nu_j^2 + [v_j]\nu_i^2), \tag{3.3}$$

where c_1, c_2 are the elastic wave velocities, ν_i^1, ν_i^2 are the normals to the wave surface:

$$\nu_1^1 = M^{-1}, \; \nu_2^1 = -M^{-1}\sqrt{M^2 - 1}, \; \nu_3^1 = \nu_3^2 = 0, \; \nu_1^2 = pM^{-1}, \; \nu_2^2 = -M^{-1}\sqrt{M^2 - p^2}.$$

Upon insertion the wedge excites two neutral strong discontinuity waves Oa and Ob moving at

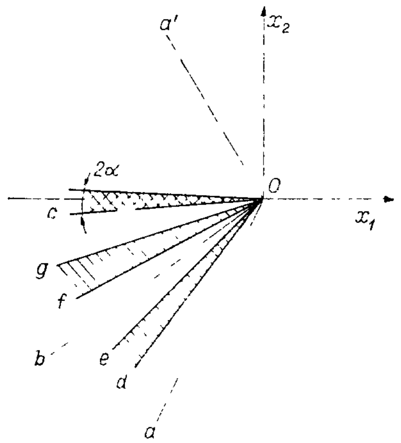

Fig. 1

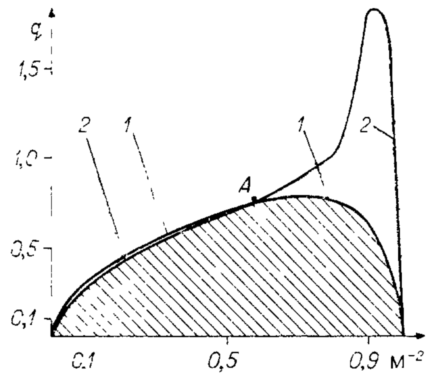

Fig. 2

the velocities c_1 anc c_2, respectively. We obtain the stress state at aOb from (3.2) in the form

$$\sigma_{11} = -\rho c_1(1 - 2p^2 + 2p^2M^{-2})\omega,\ \sigma_{22} = -\rho c_1(1 - 2p^2M^{-2})\omega,$$
$$\sigma_{12} = 2\rho c_1 p^2\sqrt{M^2 - 1}M^{-2}\omega, \qquad \sigma_{33} = -\rho c_1(1 - 2p^2)\omega. \tag{3.4}$$

We obtain at bOc from (3.3) and (3.4)

$$v_1 = \omega M^{-1} - p^{-1}[v_2]\sqrt{M^2 - p^2},\ v_2 = -\omega M^{-1}\sqrt{M^2 - 1} - [v_2],$$
$$\sigma_{11} = -\rho c_1(1 - 2p^2 + 2p^2M^{-2})\omega + 2\rho c_2M^{-1}\sqrt{M^2 - p^2}[v_2],$$
$$\sigma_{22} = -\rho c_1(1 - 2p^2M^{-2})\omega - 2\rho c_2M^{-1}\sqrt{M^2 - p^2}[v_2], \tag{3.5}$$
$$\sigma_{12} = 2\rho c_1p^2M^{-2}\sqrt{M^2 - 1}\omega - \rho c_2p^{-1}M^{-1}[v_2](M^2 - 2p^2),$$
$$\sigma_{33} = -\rho c_2(1 - 2p^2)\omega.$$

There follows from the boundary conditions (3.1) and the relationships (3.5)

$$\omega = c_1 \operatorname{tg}\alpha(M^2 - 2p^2)(M^2 - 1)^{-1/2},\ [v_2] = 2c_1p^2M^{-1}\operatorname{tg}\alpha. \tag{3.6}$$

We obtain from (3.4) and (3.6) that the medium will be in the elastic state in the domain aOb if

$$q = p^2\rho c_1^2k^{-1}\operatorname{tg}\alpha < \sqrt{M^2 - 1}\,(M^2 - 2p^2)^{-1} = r. \tag{3.7}$$

For q = r a plastic state corresponding to the edge $\sigma_1 = \sigma_3 = \sigma_2 + 2k$ will be in the domain aOb. If the inequality (3.7) is satisfied, then the medium will be in the elastic state at the domain bOc for

$$1 < M^2 < 2p^2 + 2(1 - \sqrt{2}p),$$
$$q < \sqrt{M^2 - 1}(M^2 - 2p^2)^{-1}(-1 + 2M^{-2} - 2T)^{-1}, \tag{3.8}$$

where

$$T = 2pM^{-2}(M^2 - 2p^2)^{-1}\sqrt{(M^2 - 1)(M^2 - p^2)}. \tag{3.9}$$

The curve 1, below which the inequality (3.7) is satisfied, and the curve 2 below which the inequality (3.8) is satisfied, are constructed, and intersect at the point A in Fig. 2 for $p^2 = 0.3$ in the plane (M^{-2}, q), where $q = p^2\rho c_1^2k^{-1}\operatorname{tg}\alpha$. It follows from Fig. 2 that in both the neighborhood of M = 1 and for sufficiently large M values of q will always exist for which an elastic solution will be impossible. The elastic solution is possible only in the domain below both curves (shaded domain in Fig. 2).

Analysis of the inequalities (3.7)-(3.9) shows that for $p^2 < 1/4$ as well as for $p^2 \geqslant 1/4$, $4p^4(4p^2 - 1)^{-1} \geqslant M^2 \geqslant 2p^2 + 2(1 - \sqrt{2p})$ the plasticity is first manifest in the domain aOb as tg α increases, if $q = \sqrt{M^2 - 1}(M^2 - 2p^2)^{-1}$. The maximal tangential stress achieved the value k in the domain bOc only for $p^2 > 1/4$, $M^2 \geqslant 4p^4(4p^2 - 1)^{-1}$ as well as for $p^2 < 1/2$ and $1 \leqslant M \leqslant 2p^2 + 2(1 - \sqrt{2p})$. As tg α increases the plasticity is first manifest if

$$q = \sqrt{M^2 - 1}(M^2 - 2p^2)^{-1}(1 - 2M^{-2} + 2T)^{-1},\quad = \sqrt{M^2 - 1}(M^2 - 2p^2)^{-1}(-1 + 2M^{-2} - 2T)^{-1}$$

respectively. We obtain from (3.5) for normal pressure on the wedge face

$$\sigma_{22} = -\rho c_1^2 M^{-2}\left[4p^3\sqrt{M^2-p^2} + (M^2-2p^2)^2(M^2-1)^{-1/2}\right]\operatorname{tg}\alpha.$$

Let the medium in the domain bOc be in the plastic state, and in the elastic state in aOb, then the solution in aOb is determined from formula (3.4), in bOf from (3.5), plastic strain occurs in fOg, and the integrals (2.2), (2.4), (2.5) are executed, the stresses and displacement velocities are constant in gOc and equal to their values on the line Og. We obtain $\theta = 0$ on the line Og from (3.1) and we have the following equation from (2.2)

$$p^2(1-p^2)\cos^2 2\varphi_1 = M^2\sin^2\varphi_1(1-M^2\sin^2\varphi_1) \tag{3.10}$$

to determine φ_1 (the slope of Og to the x_1 axis).

Equation (3.10) has the solution

$$\sin^2\varphi_1 = \frac{M^2+4p^2(1-p^2) - M\sqrt{M^2(1-2p^2)^2+8p^2(1-p^2)}}{2M^4+8M^2(1-p^2)} \tag{3.11}$$

in bOc where $M^2\sin\varphi \leqslant p^2$. Plastic strain occurs in bOf, from (1.6) and (3.5) there follows

$$\omega^2p^2 + [v_2]^2M^2 - 4p^2M^{-3}\omega[v_2]\left(\sqrt{M^2-1}(M^2-2p^2) - p(M^2-2)\sqrt{M^2-p^2}\right) = k^2p^{-2}(\rho c_1^2)^{-1}. \tag{3.12}$$

On the line Of where $\varphi = \varphi_2$ the following nontrivial solution holds

$$p^2(1-p^2)\cos^2 2(\varphi_2-\theta_2) = (1-M^2\sin^2\varphi_2)M^2\sin^2\varphi_2. \tag{3.13}$$

From the condition of continuity of σ_{12} we obtain on the line Of

$$2\rho c_1p^2\sqrt{M^2-1}M^{-2}\omega - \rho c_1(M^2-2p^2)M^{-1}[v_2] = k\sin 2\theta_2. \tag{3.14}$$

For $\varphi = \varphi_2$ we have $v_2(\varphi_2) = -[v_z] - \omega M^{-1}\sqrt{M^2-1}$ from (3.6); consequently there follows from (2.8) and (3.1) the relationship

$$\begin{aligned}&\operatorname{sign}(\sin 2(\varphi-\theta))\Big[2\sqrt{(p^2-M^2\sin^2\varphi)(1-p^2-M^2\sin^2\varphi)} - \\ &-(1-2p^2)\ln\left|\sqrt{p^2-M^2\sin^2\varphi}+\sqrt{1-p^2-M^2\sin^2\varphi}\right|\Big] + \\ &+(2p^2-M^2)\ln\left|M\cos\varphi+\sqrt{1-M^2\sin^2\varphi}\right| + 2p\sqrt{M^2+p^2-1}\times \\ &\times\ln\left|\frac{M\cos\varphi-\sqrt{(M^2+p^2-1)(1-M^2\sin^2\varphi)}}{M\cos\varphi+\sqrt{(M^2+p^2-1)(1-M^2\sin^2\varphi)}}\right| + 2M\cos\varphi\sqrt{1-M^2\sin^2\varphi} + \\ &+[v_2]+\omega\sqrt{M^2-1}M^{-1} = c\operatorname{tg}\alpha.\end{aligned} \tag{3.15}$$

For given values of $\operatorname{tg}\alpha$ values of ω, $[v_2]$, φ_2, θ_2 are determined from the relationships (3.12)-(3.15). This solution holds while $|\omega|p^2\rho c_1 \leqslant k$. For $\omega = k(p^2\rho c_1)^{-1}$ the yield point is reached in the domain aOb. Substituting the value ω obtained into (3.12) and (3.15), we obtain the maximal values of q for which this solution is still possible. We limit ourselves to the solution of (3.12)-(3.15) for large M. Here $\sin^2\varphi_1 = p^2M^{-2} - 4p^4(1-p^2)(1-2p^2)^{-1}\cdot M^{-4} + o(M^{-6})$. The maximal q for which the medium will be in the elastic state in aOb while plastic strain occurs in bOc in the zone fOg, will have the form

$$q \leqslant M^{-2}\sqrt{M^2-1}. \tag{3.16}$$

Setting $\sin^2\varphi_2 = p^2M^{-2} - bM^{-4}$, where b is an unknown quantity, we have from (3.12)-(3.15)

$$b = p^2(1-p^2)(1-2p^2)^{-1}[2p - qM^3(M^2-4p^2)^{-1}]^2.$$

The normal pressure on the wedge face is expressed in the form

$$\sigma_{22} = -k - (1-p^2)\rho c_1^2M^2(M^2-1)^{-1/2}\operatorname{tg}\alpha + kM^{-2}p^{-2}(a-b),$$

where

$$a = 4p^4(1-p^2)(1-2p^2)^{-1}.$$

Let the medium be in a plastic strain state in the domain aOb, and in the elastic state in the domain bOc. Then the solution in the zone eOd is determined from (2.6)-(2.9), and from (3.5) in bOc, where the stresses and displacement velocities are constant in bOc and equal to the values directly behind the line Ob. In aOd the stresses and displacement velocities are determined from (3.4), where the intensity of the irrotational wave is $\omega = kp^{-2}(\rho c_1)^{-1}$. For media with $p^2 > 3/7$ the allowable values of M from (2.7) are small. If the medium is characterized by the values $1/4 < p^2 < 3/7$ deformation is possible in the mentioned form for large M.

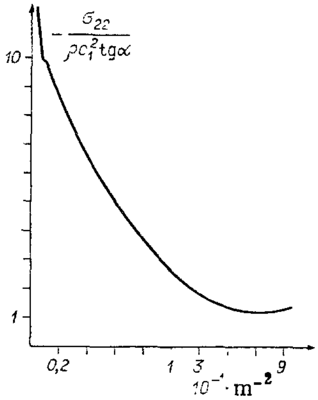

Fig. 3

The maximal value of q obtained under the assumption of attainment of the plasticity state beyond the line Ob, where deformation of the type mentioned is still possible, is expressed in the form

$$q \leqslant M^{-2}\sqrt{M^2-1}+\left(1-\frac{7}{3}p^2\right)\left(\ln\frac{r}{a}\right)\left(\sqrt{1-\frac{4}{3}p^2}\,2M\right)^{-1}, \tag{3.17}$$

where

$$M^2\sin^2\varphi_1 = 1-\frac{4}{3}p^2+rM^{-2},$$

$$M^2\sin^2\varphi_4 = 1-\frac{4}{3}p^2+aM^{-2} = 1-\frac{4}{3}p^2+4p^2\left(1-\frac{4}{3}p^2\right)\left(1-\sqrt{1-\frac{4}{3}p^2}\right)^2\left(1-\frac{7}{3}p^2\right)^{-1}M^{-2}.$$

An equation to determine r follows from the boundary condition (3.1)

$$q = M^{-2}\sqrt{M^2-1}+\left(1-\frac{7}{3}p^2\right)\left(2M\sqrt{1-\frac{4}{3}p^2}\right)^{-1}\ln\frac{r}{a}+$$
$$+p^2M^{-1}(M^2-2p^2)^{-1}\left[2\sqrt{1-\frac{4}{3}p^2}-\sqrt{r\left(1-\frac{7}{3}p^2\right)\left(1-\frac{4}{3}p^2\right)^{-1}p^{-2}}\right]+$$
$$+3p\sqrt{1-\frac{7}{3}p^2}\left(1-\frac{4}{3}p^2\right)^{-1/2}M^{-3}+\frac{1}{4}\left[(r-a)\left(1+\frac{5}{2}p^2\right)p^{-2}-\right.$$
$$\left.-\left(1-\frac{7}{3}p^2\right)\left(1-\frac{4}{3}p^2\right)\ln\frac{r}{a}\right]\left(1-\frac{4}{3}p^2\right)^{-1/2}M^{-3}.$$

The normal pressure on the wedge face is expressed in the form

$$\sigma_{22} = -\frac{k}{p^2}-\frac{k}{p^2}\left(1-\frac{7}{3}p^2\right)\ln\frac{r}{a}+o\left(\frac{1}{M^2}\right).$$

If the inequality (3.16) or (3.17) is not satisfied, then in both the domain aOb and the domain bOc the medium will be in the plasticity state.

The solutions in aOd are determined by (3.4), while plastic deformation occurs in dOe and the integrals (2.2)-(2.5) hold, and the stresses and displacement velocities are constant in dOb and equal to their values on the line Oe.

The solutions in the zone bOf are determined by (3.5), the integrals (2.2)-(2.5) hold in fOg, and the stresses and displacement velocities retain their values behind the line Og, equal to their values on the line Og. For large values of M we use the notation

$$M^2\sin^2\varphi_1 = p^2-4p^2\left(1-\frac{4}{3}p^2\right)\left(1-\frac{7}{3}p^2\right)^{-1}M^{-2} = p^2-hM^{-2},$$

$$M^2\sin^2\varphi_2 = p^2-eM^{-2},\quad M^2\sin^2\varphi_3 = 1-\frac{4}{3}p^2+rM^{-2},$$

$$M^2\sin^2\varphi_4 = 1-\frac{4}{3}p^2+4p^2\left(1-\frac{4}{3}p^2\right)\left(1-\frac{7}{3}p^2\right)^{-1}\left(1-\sqrt{1-\frac{4}{3}p^2}\right)^2M^{-2} = 1-\frac{4}{3}p^2+aM^{-2}.$$

The plastic deformation in zones dOe and fOg occur under the identical condition (1.18); therefore, the following equality holds

$$\rho c_2[v_2] = 2kp^{+2}M^{-2}[M^{-1}p^{-1}(M^2-2p^2)\sin 2\theta_3-2M^{-1}\sqrt{M^2-p^2}\cos 2\theta_3].$$

From the condition of continuity of σ_{12} we obtain on the line Of

$$e+r=4p^2\left(1-\frac{4}{3}p^2\right)\left(1-\frac{7}{3}p^2\right)^{-1}\left(\sqrt{1-\frac{4}{3}p^2}-p\right)^2.$$

There follows from the boundary condition (3.1)

$$\ln\frac{r}{a}=-2Mq\sqrt{1-\frac{4}{3}p^2}\left(1-\frac{7}{3}p^2\right)^{-1}.$$

The normal pressure on the wedge face is expressed in the form

$$\sigma_{22}=-\frac{k}{p^2}+\frac{k}{p^2}\left(1-\frac{7}{3}p^2\right)\ln\frac{r}{a}+o\left(\frac{1}{M}\right).$$

The pressure change on the wedge face is shown in Fig. 3 for $p^2 = 0.3$ and $q = 0.1$ as a function of the velocity of wedge penetration. We obtain the minimal pressure in the elastic solution for $M^2 = 1.18$. As M decreases and increases the pressure rises. At the value $M^2 = 99.82$ plastic flow sets in at the wedge face (the domain bOc in Fig. 1) and the pressure growth is terminated. For $M^2 = 100.20$ plasticity sets in even in the domain aOb. Later the pressure grows in proportion to M as M increases.

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THEORY OF IDEAL PLASTICITY OF MULTICOMPONENT MIXTURES

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UDC 539.4

1. We consider a rigidly plastic micro-inhomogeneous isotropic medium consisting of n different components interconnected by ideal adhesion. Let the plastic properties of each component be described by the surface flow taking the hydrostatic pressure into account

$$s_{ij}s_{ij}+a_s\sigma_{pp}^2=k_s^2,\quad s=1,2,\ldots,n,$$

where $s_{ij} = \sigma_{ij} - \delta_{ij}\sigma_{pp}/3$, σ_{ij} is the stress tensor, k_s are the component yield points, and a_s are parameters characterizing their volume compressibility.

The structure of such a medium can be described by a system of random indicator functions of the coordinates $\varkappa_1(\mathbf{r}), \varkappa_2(\mathbf{r}), \ldots, \varkappa_n(\mathbf{r})$, from which each function $\varkappa_s(\mathbf{r})$ equals unity on a set of points of the s-th component and equals zero outside this set. By using these functions the local associated flow law of the composite material under consideration can be written in the form [1]

$$\sigma_{ij}(\mathbf{r})=k(\mathbf{r})\frac{\varepsilon_{ij}(\mathbf{r})-\delta_{ij}b(\mathbf{r})\varepsilon_{pp}(\mathbf{r})}{\sqrt{\varepsilon_{kl}(\mathbf{r})\varepsilon_{kl}(\mathbf{r})-b(\mathbf{r})\varepsilon_{pp}^2(\mathbf{r})}},\tag{1.1}$$

where $\varepsilon_{ij}(\mathbf{r})$ is the strain rate tensor; $k(\mathbf{r})=\sum_{s=1}^{n}k_s\varkappa_s(\mathbf{r})$; and

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